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FAST TRACK COMMUNICATION

Importance of the Wick rotation on tunnelling

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Abstract

A continuous complex rotation of time $t \mapsto t e^{-i\theta}$ is shown to smooth out the huge fluctuations that characterize chaotic tunnelling. This is illustrated in the kicked rotor model (quantum standard map) where the period of the map is complexified: the associated chaotic classical dynamics, if significant for $\theta = 0$, is blurred out long before the Wick rotation is completed ($\theta = \pi/2$). The influence of resonances on tunnelling rates weakens exponentially as θ increases from zero, all the more rapidly the sharper the fluctuations. The long-range fluctuations can therefore be identified in a deterministic way without ambiguity. When the last ones have been washed out, tunnelling recovers the (quasi-)integrable exponential behaviour governed by the action of a regular instanton.

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(Some figures in this article are in colour only in the electronic version)

It is often admitted that the Wick rotation $t \rightarrow -it$ provides a straightforward route from a Minkowskian metric to an Euclidean metric or from a zero-temperature quantum model to a finite temperature statistical model. In this communication we will show that even a small complex rotation of time, $t \rightarrow e^{-i\theta}t$ can affect drastically the spectral properties of some operators that encapsulate the quantum dynamics. In particular, the *fluctuations* of tunnelling rates and therefore some quantum transport properties, can change by several order of magnitude even if $\theta \ll 1$.

This work is especially motivated by the topic of chaotic tunnelling, a theoretically challenging subject that has been resisting a systematic and satisfying solution for more than 16 years. Since the pioneer quantitative studies (Lin and Ballentine 1990), it has been widely observed¹ that tunnelling, i.e. a quantum process that is forbidden at a classical level,

¹ Mainly numerically and to a lesser extent experimentally (Mouchet *et al* 2001, Hensinger 2001, Steck *et al* 2001, Hensinger *et al* 2004), see also the discussion in (Mouchet and Delande 2003).

exhibits a huge sensitivity to any perturbation and generically fluctuates by several orders of magnitudes if the underlying classical dynamics is non-integrable. Understanding, predicting and controlling this behaviour remain widely open problems for which several strategies have been proposed. Following semiclassical methods, known to be successful in tackling the issues of tunnelling in integrable systems, the most natural strategy is to try to express a tunnelling rate (for instance a decay time or an oscillation period between two symmetric wells) in terms of complex solutions of Hamilton's equations only. Already highly non-trivial in the multidimensional (non-separable) integrable or quasi-integrable cases (Miller 1974, Wilkinson 1986), this programme appears even more difficult when some of the integrals of motions are strongly broken. Indeed, it has been discovered (Shudo and Ikeda 1995) that chaos reveals itself in the complex phase space through some fractal structures, the so-called Laputa islands, that look like agglomerates of complex classical trajectories. It is only recently that some encouraging, significant steps were realized in retaining the relevant semiclassical skeleton (Shudo *et al* 2002) for tunnelling. To bypass this purely semiclassical strategy, it has been proposed (Bohigas *et al* 1993) to replace the chaotic, though deterministic, transition amplitudes (i.e. involving one or more chaotic quantum states) by some random matrix elements and hopefully compute an average tunnelling behaviour on the appropriate statistical ensemble. This hybrid approach, mixing together some semiclassical integrable ingredients and statistical chaotic ones, allows us to establish a precise and quantitative connection between the quantum and classical resonances which play a crucial rôle for understanding multidimensional tunnelling (Brodier *et al* 2001). When classical chaos is well developed, the overlap of classical resonances has a quantum counterpart: the collective effect of the quasi-coincidences of quantum frequencies, which cannot be isolated one from the other, not only causes the huge fluctuations of a tunnelling rate but also enhances its average behaviour (Eltschka and Schlagheck 2005, Schlagheck *et al* 2006, Mouchet *et al* 2006). In this context, the complex rotation of time that is proposed in this communication provides a continuous way to understand how the resonances conspire to create a chaotic tunnelling regime. No statistical ingredients are required and therefore the present approach establishes a new bridge between the two strategies described above since it provides a deterministic, resonances-governed transition between a regular and a chaotic tunnelling regime where the complex classical solutions play a natural and essential role.

One of the simplest models where chaotic tunnelling is at work corresponds to a quantum system with one degree of freedom whose dynamics is a sequence of periodically alternating kinetic and potential motions (Berry *et al* 1979). After one period τ , the quantum evolution operator (also known as the quantum map or the Floquet operator) is given by

$$\hat{U}(\tau) = e^{-i\tau f(\hat{p})/\hbar} e^{-i\tau g(\hat{q})/\hbar}. \quad (1)$$

The associated classical dynamics after one period is described by the discrete Hamilton equations (the Poincaré map): $(p_0, q_0) \mapsto (p_1, q_1)$ with $p_1 = p_0 - \tau g'(q_0)$ and $q_1 = q_0 + \tau f'(p_1)$ (the primes stand for the derivative of the smooth functions f and g). The choice $f(p) \stackrel{\text{def}}{=} p^2/2$ and $g(q) \stackrel{\text{def}}{=} \gamma \cos q$, where γ is a real parameter, corresponds to the well-known kicked rotor model (the standard map) that is extensively studied for understanding the subtle interplay between classical and quantum transport (Casati *et al* 1979). It is convenient to work with the usual symmetric double-well situation that can be obtained from the kicked rotor model by unfolding the cylindrical phase space on a double spatial period and work in the following with $g(q) \stackrel{\text{def}}{=} \gamma \cos 2q$ with strictly periodic quantum periodicity for any q -translation by 2π .² The classical dynamics is shown in figure 1. No real trajectories connect

² It is of course equivalent of retaining the q -antiperiodic as well as the q -periodic quantum states for the standard quantum map.

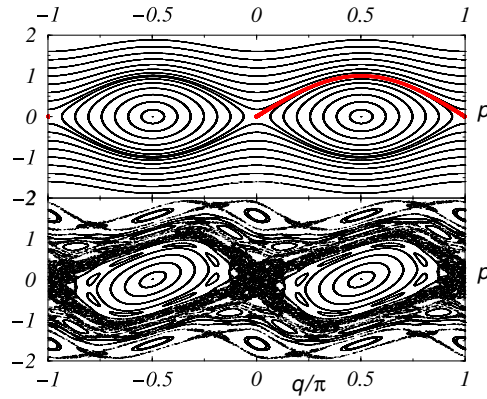


Figure 1. Real phase-space trajectories for the kicked rotor with τ real and $\gamma = 0.25$. The upper panel corresponds to the quasi-integrable case ($\tau \rightarrow 0^+$). The red thick line is the trajectory joining the unstable fixed point ($p = 0, q = 0$) to ($p = 0, q = \pi$) after an infinite number of iterations of the map. The mixed dynamics in the lower panel corresponds to $\tau = 1$ where a chaotic sea separates the two regular islands in the neighbourhood of the stable points ($p = 0, q = \pm\pi/2$).

the interior of the two symmetric islands in the neighbourhood of the two fixed points $p = 0$ and $q = \pm\pi/2$ that remain stable as long as

$$\tau \in]-1/\sqrt{\gamma}, 1/\sqrt{\gamma}[. \quad (2)$$

At the quantum level, we will consider the dynamics that is described in terms of the eigenstates of (1) that are strictly invariant under a q -translation by 2π . The classical phase-space parity symmetry $(p, q) \mapsto (-p, -q)$ allows us to classify the spectrum of (1) accordingly: we will denote u_n^+ (resp. u_n^-) as the eigenvalues corresponding to the symmetric (resp. antisymmetric) eigenstates $|\phi_n^\pm\rangle$, where n is a quantum number labelling the doublet. For real τ , the quasi-energies defined modulo $2\pi\hbar/\tau$ by $\epsilon_n^\pm \stackrel{\text{def}}{=} \frac{i\hbar}{\tau} \ln(u_n^\pm)$ is real. For a given n , when $|\phi_n^\pm\rangle$ have their Husimi representation localized inside the two wells, tunnelling is characterized by the splitting $\Delta\epsilon_n \stackrel{\text{def}}{=} \epsilon_n^- - \epsilon_n^+$. In the following, we will drop the n , keeping in mind that we will work with the central doublet i.e., selected by the criterion of having the maximal overlap with a coherent state that is located on the stable fixed points³. The expected semiclassical behaviour of $\Delta\epsilon$ in a (quasi-)integrable regime where no classical resonance can be resolved by quantum eyes is given by Wilkinson (1986)

$$\Delta\epsilon \underset{\hbar \rightarrow 0}{\sim} \alpha \hbar^\nu e^{-|A|/\hbar}, \quad (3)$$

where α and A are made of classical (\hbar -independent) ingredients. The exponent ν depends on the nature (integrable or quasi-integrable) of the dynamics. For a $1D$ time-independent system, it is well known (Landau and Lifshitz 1977) that $\nu = 1$ and both α and $|A|$ can be interpreted in terms of instantons (Mclaughlin 1972), i.e. expressed from the real trajectory joining (in an infinite time) the two fixed points once the Wick rotation ($\theta = \pi/2$) has been completed. In multidimensional systems or in the case of a quantum map, the generic presence of quantum (and classical) resonances will drastically modify (3). This can be understood as follows: for τ real, the u 's accumulate on the unit circle and two quasi-energies differing by almost an integer multiple of $2\pi\hbar/\tau$ will make the corresponding u 's almost coincide and

³ This is not essential: it has been checked that the general feature described in this paper applies also to the 'first excited' doublet.

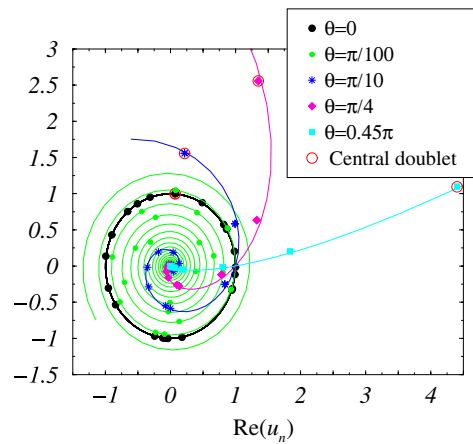


Figure 2. (Colour online) Eigenvalues u_n^\pm of (1) in the complex plane for $\gamma = 0.25$, $\hbar \simeq 1/8.002$ and, $\tau = e^{-i\theta}$ with $\theta = 0$: large dots (black); $\theta = 10^{-2}\pi$: small dots (green); $\theta = \pi/10$: stars (blue); $\theta = \pi/4$: diamonds (magenta); $\theta = 0.45\pi$: squares (cyan). The tunnelling central doublet is circled (red). Its splitting is of order at most 10^{-5} (see figure 3) and therefore cannot be resolved at this scale. The continuous lines are the logarithmic spirals $e^{-i\tau s/h}$ parametrized by the real s .

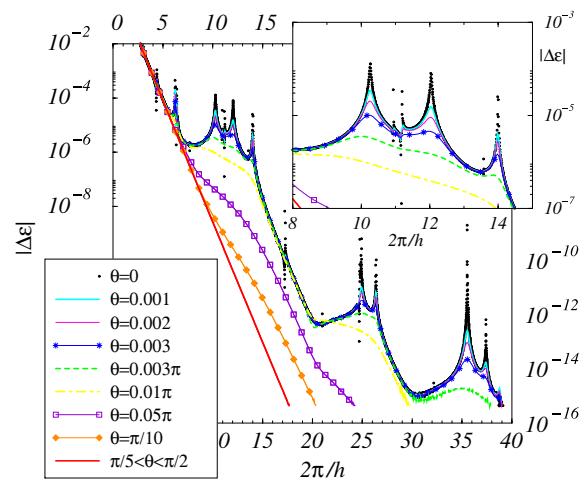


Figure 3. (Colour online) Modulus of the central tunnelling splitting for the kicked rotor with $\gamma = 0.25$, $\tau = e^{-i\theta}$, $\theta = 0$: dots (black); $\theta = 10^{-3}$: thick solid line (cyan); $\theta = 2 \times 10^{-3}$: thin solid line (magenta); $\theta = 3 \times 10^{-3}$: starred solid line (blue); $\theta = 3 \times 10^{-3}\pi$: dotted line (green); $\theta = 10^{-2}\pi$: dashed line (yellow); $\theta = 5 \times 10^{-2}\pi$: squared line (violet); $\theta = \pi/10$: diamond line (orange); $\pi/5 \leq \theta \leq \pi/2$: thick solid line (red).

generate small denominators in any perturbative expansion. Some transitions between the central doublet and some excited doublets are therefore greatly enhanced and give birth to large fluctuations when a control parameter is varied. In figure 3, the latter is chosen to be the inverse of the (effective) Planck constant and $\Delta\epsilon$ is plotted in black dots for $\tau = 1$ and $\gamma = 0.25$ corresponding to the classical dynamics in the lower panel in figure 1.

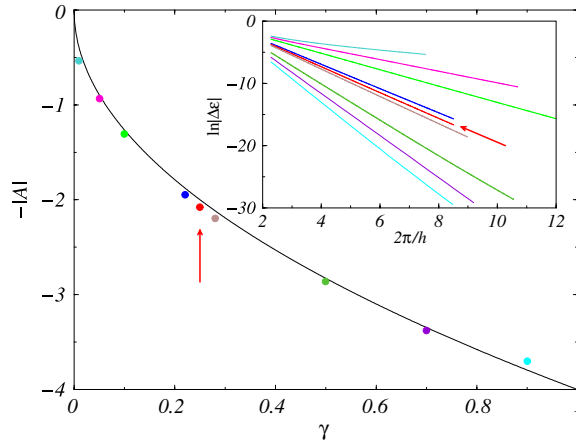


Figure 4. (Colour online) As θ increases, the semi-logarithmic graphs of $1/\hbar \mapsto |\Delta\epsilon|$ eventually accumulate on straight lines (shown in the inset) whose slope $-|A|$ (see (3)) is reported for several values of γ ($|\tau| = 1$ being fixed). The red arrows indicate the case $\gamma = 0.25$ corresponding to figures 1 and 3. The continuous black line is the graph of $-4\sqrt{\gamma}$.

The operator $\hat{U}(\tau)$ can be analytically continued in the lower half-plane of complex τ 's. Giving a negative imaginary part to τ makes \hat{U} non-unitary and its eigenvalues escape from the unit circle into the whole complex plane. In the quasi-integrable case obtained for small enough $|\tau|$ and γ for the two exponentials in (1) almost commute, it is expected that the eigenvalues u_n^\pm remain near the logarithmic spiral $e^{-i\tau s/\hbar}$ parametrized by the real s . It is surprisingly the case even in a regime where classical chaos is well developed (for $|\tau| = 1, \gamma = 0.25$, see figure 1) and τ being far from the real axis. A complex rotation of τ by a positive angle $\theta \stackrel{\text{def}}{=} -\arg \tau$ appears to be a simple tool to unfold the unit circle into a spiral and therefore to move the resonant doublets away from each other: an increase of the quasi-energy by $2\pi\hbar/|\tau|$ is accompanied by a shrinking of the modulus of the u 's by a factor of $\exp(-2\pi \sin \theta)$. The larger the θ , the less the influence of the resonances on the tunnelling splitting. This is confirmed by the plots in figure 3 where we can see that, not only an increase of θ from 0 induces a systematic decrease of chaotic tunnelling rate, but also that the fluctuations of $|\Delta\epsilon|$ are progressively smoothed by a complex time rotation. For θ 's about 10^{-3} , only the more acute spikes are eroded. For $\theta \simeq 0.01\pi$, only the long-range fluctuations in $1/\hbar$ have survived and the staircase-like structure that can be observed on the semi-logarithmic plot is very similar to the average behaviour of $\ln|\Delta\epsilon|$ that is obtained from the resonant-assisted tunnelling hybrid method described above (compare with Eltschka and Schlagheck (2005), figures 1 and 2 and Mouchet *et al* (2006), figure 10) but here no averaging process is required and there is no rough transition between staircase steps due to the semiclassical truncation of the Hamiltonian between a regular part and the random matrix that models the transitions involving a chaotic state. What is unexpected and of course requires further future investigations, is that the splitting curves (in semi-log plots) converge quickly, say for $\pi/5 \lesssim \theta \leq \pi/2$, to a straight line in agreement with the (quasi-)integrable behaviour (3). The slope $-|A|$ is roughly the same as the one that appears in between the plateaus for smaller θ . Moreover, keeping $|\tau| = 1$ and computing the slope of the limiting straight lines obtained for several values of γ , including the ones for which the stable islands are almost dissolved in the chaotic sea for real τ (according to (2) the bifurcation point where the stable fixed points lose their stability occurs at $\gamma = 1$). It is shown in figure 4 that $|A| \simeq 4\sqrt{\gamma}$,

which can be interpreted as the action of the instanton of the regular regime. It is easy to check that the complete Wick rotation ($\theta = \pi/2$) of the kicked rotor corresponds to a simple q -translation by $\pi/2$ of the dynamics obtained for $\theta = 0$. Therefore the action of the instanton that joins ($p = 0, q = \pm\pi$) in an infinite purely imaginary time is the area $4\sqrt{\gamma}$ under the separatrix shown in the upper panel of figure 1. From a purely semiclassical point of view, as recalled above, one needs to elucidate the relevant structures in the complex phase space. For $\theta > 0$, those are expected to be more tractable, compared to the situation where $\theta = 0$. It can be predicted without too much risk that the inextricable fractal structure of the Laputa islands (for the kicked rotor see figure 2 of Shudo *et al* (2002)) simplifies when increasing θ . In other words, decreasing θ towards 0 should allow us to understand the formation of the delicate pattern of the Laputa chains from isolated orbits in the same time as we see in figure 3 how the resonances overlap and glue together in a dense set of $\Delta\epsilon$ fluctuations.

The above analysis raises of course several (still open) questions. In particular the rôle of the regular instanton is puzzling since it represents a primitive classical trajectory with infinite period and therefore seems out of reach, if one were to tentatively try to obtain a trace formula à la Gutzwiller starting from the estimate

$$\Delta\epsilon_0 \sim \frac{2i\hbar \operatorname{tr}(\hat{S}\hat{U}^N(\tau))}{N\tau \operatorname{tr}(\hat{U}^N(\tau))} = \frac{2i\hbar \sum_n (e^{-iN\tau\epsilon_n^+/\hbar} - e^{-iN\tau\epsilon_n^-/\hbar})}{N\tau \sum_n (e^{-iN\tau\epsilon_n^+/\hbar} + e^{-iN\tau\epsilon_n^-/\hbar})} \quad (4)$$

(\hat{S} is the parity operator) valid in a regime where the integer N is sufficiently large for the central doublet to dominate all the others but sufficiently small so that $N\tau\Delta\epsilon/\hbar \ll 1$. Indeed, the usual resummation techniques lead to an expansion arranged in ascending order of the length of the *primitive* (pseudo-)periodic⁴. As far as tunnelling is concerned, I have not been able to extract any relevant information from the shortest complex pseudo-periodic orbits. The chaotic character, that seems to fade away for $0 < \theta < \pi/2$, must also be understood better. In this interval, all the orbits that are stable for $\theta = 0$ or $\pi/2$ get generically destabilized since the traces of the monodromy matrices escape from $]-2, 2[$ into the complex plane but the mixing properties in a non-compact complex phase space may vanish as well. The non-vanishing $\operatorname{Im}\tau$ has the flavour of a dissipation, and the suppression of the chaos-enhancement of tunnelling reminds of the Caldeira–Leggett model (Caldeira and Leggett 1981).

The continuous complex rotation of time undoubtedly opens new promising perspectives for gaining a proper understanding of chaotic tunnelling and, as recalled in the introduction, this new approach sheds light on the subtle non-analytical connection that may happens concerning the transport properties of complex systems that are linked by the Wick rotation.

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⁴ The expansions of the denominator and the numerator of (4) combine into pseudo-periodic orbits made from the concatenation of strictly periodic orbits with period $N\tau$ (denominator) with one trajectory of length $N\tau$ connecting two points related by parity (numerator).

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